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Pointed weak energy and quantum geometric phase

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Abstract
The pointed weak energy for an arbitrarily evolving quantum state defines a complex valued phase which is the sum of a dynamic phase and a purely geometric phase—the pointed geometric phase. The pointed geometric phase is concisely expressed as a time integral which depends upon the energy uncertainty, the associated evolving state, and its orthogonal companion state. The real part of the pointed geometric phase is to within a sign the geometric phase for arbitrary evolutions defined by Mukunda and Simon and that of Aharonov and Anandan for cyclic evolutions. The imaginary part of the pointed geometric phase governs the survival probability of the initial state. Several general rate of change relationships associated with the real and imaginary parts of the pointed geometric phase are deduced from this concise expression, and it is used to calculate the pointed geometric phase acquired as a spin-$\frac{1}{2}$ particle precesses under the influence of a uniform magnetic field.

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1. Introduction

A half century ago, S Pancharatnam discovered that if a series of polarizers constrain the polarization state of light to follow a closed three cycle along geodesic arc segments in polarization state space (i.e., the Poincare′ sphere), then the state acquires a geometric phase equal to half the solid angle subtended by the associated geodesic triangle [1]. Three decades later, Berry showed that the phase acquired by a quantum system during an adiabatic cyclic evolution is the sum of a term proportional to the time integral of the instantaneous eigenenergy and a term proportional to an integral over the cycle in system parameter space described by the slowly varying parameter-dependent Hamiltonian (the geometric Berry phase) [2]. A subsequent generalized reformulation of this by Aharonov and Anandan demonstrated that the phase change during any cyclic evolution of a quantum system can be partitioned into the sum of a dynamical phase that is proportional to the time integral of the instantaneous mean value of the system Hamiltonian and a geometric phase that depends only upon the cycle traced in state space by the evolving state [3]. These geometric phases have been observed and measured experimentally, e.g. [4, 5], and the theory has been extended to describe the geometric phase associated with arbitrary evolutions of quantum systems, e.g. [6, 7].

Recently a complex valued *pointed weak energy* that is associated with the Hilbert space evolution of a quantum state relative to its initial state has been defined and studied from a theoretical perspective [8, 9]. It was shown that the pointed weak energy defines an exponential function which acts upon such an evolving state's initial correlation amplitude and translates it in time. The argument of this exponential function is a complex valued phase that is acquired by the evolving amplitude relative to its initial unit value. It is shown in this paper that this phase is the sum of a dynamical phase and a purely geometric complex phase—the *pointed geometric phase*. The real part of the pointed geometric phase is—to within a sign—the geometric phase for arbitrary evolutions defined by Mukunda and Simon (the MS phase) [10] and—consequently—is also the Aharonov–Anandan (AA) phase for cyclic evolutions. The imaginary part of the pointed geometric phase defines the modulus of the correlation amplitude and therefore governs the survival probability of the initial quantum state.

The pointed geometric phase is concisely expressed as a time integral—the integrand of which depends upon the energy uncertainty, the evolving state, and its orthogonal companion state. Several interesting general rate of change relationships associated with the real and imaginary parts of the pointed geometric phase, i.e. phase s-speeds and the s-speed ratio, are deduced from this expression. As an illustrative example, this concise expression is used to calculate the pointed geometric phase acquired as a spin-$\frac{1}{2}$ particle precesses under the influence of a uniform magnetic field. Time-dependent expressions for the phase s-speeds and the s-speed ratio are also obtained and briefly discussed for this spin-$\frac{1}{2}$ particle system.

2. Pointed weak energy and time translation

Let $|\psi(t)\rangle$ be a normalized quantum state evolving in a Hilbert space $\mathcal{H}$ with projective space $\mathcal{P}$ consisting of all the rays of $\mathcal{H}$ (recall that a ray is an equivalence class $[\psi]$ of states $|\psi\rangle$ in $\mathcal{H}$ which differ only in phase) and with the induced projection map $\Pi : \mathcal{H} \rightarrow \mathcal{P}$ such that $|\psi\rangle \stackrel{\Pi}{\mapsto} [\psi]$. The *pointed weak energy* $W_0(t)$ associated with this state is the complex valued quantity defined by [8]

$$W_0(t) \equiv \frac{\langle\psi(t)|\widehat{H}|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle} = \mathrm{Re}\, W_0(t) + \mathrm{i}\, \mathrm{Im}\, W_0(t), \tag{1}$$

where the correlation amplitude appearing in the denominator of this expression satisfies $\langle\psi(t)|\psi(0)\rangle \neq 0$, and the state $|\psi(t)\rangle$ evolves according to

$$\mathrm{i}\hbar \frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \widehat{H}|\psi(t)\rangle.$$

It is shown in [8] that

$$\mathrm{Re}\, W_0(t) = \hbar \left(\frac{\mathrm{d}\chi(t)}{\mathrm{d}t}\right) \equiv \hbar\dot{\chi}(t) \tag{2}$$

and

$$\mathrm{Im}\, W_0(t) = \hbar \left\{\frac{s(t)}{4 - s^2(t)}\right\} \left(\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right) \equiv \hbar \left\{\frac{s(t)}{4 - s^2(t)}\right\} \dot{s}(t). \tag{3}$$

Here, $\chi(t)$ is the Pancharatnam phase at time t defined by

$$\mathrm{e}^{\mathrm{i}\chi(t)} = \frac{\langle\psi(t)|\psi(0)\rangle}{|\langle\psi(t)|\psi(0)\rangle|} \tag{4}$$

and is the phase difference between $|\psi(0)\rangle$ and $|\varphi\rangle$, where $|\varphi\rangle$ is the state contained in the equivalence class $[\psi(0)]$ obtained by parallel transporting $|\psi(t)\rangle$ along the shortest geodesic joining $[\psi(t)]$ and $[\psi(0)]$ in $\mathcal{P}$ [11]. The function $s(t)$ is the distance separating $[\psi(t)]$ and $[\psi(0)]$ in $\mathcal{P}$ at time t given by the generalized Fubini–Study metric defined by [12, 13]

$$s^2(t) \equiv 4(1 - |\langle\psi(t)|\psi(0)\rangle|^2). \tag{5}$$

Also shown in [8, 9] is the property that the pointed weak energy defines an exponential function which time translates the initial state correlation amplitude at time $t = 0$ to its new value at time $\tau > 0$ via

$$\langle\psi(\tau)|\psi(0)\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^\tau W_0(t)\mathrm{d}t}\langle\psi(0)|\psi(0)\rangle,$$

or—more precisely—since $\langle\psi(0)|\psi(0)\rangle = 1$ and equation (1) both apply, it defines the amplitude at time τ according to

$$\langle\psi(\tau)|\psi(0)\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\int_0^\tau \mathrm{Re}\, W_0(t)\mathrm{d}t - \frac{1}{\hbar}\int_0^\tau \mathrm{Im}\, W_0(t)\mathrm{d}t}. \tag{6}$$

3. The pointed geometric phase

Consider the fact that the action of the Hamiltonian operator $\widehat{H}$ upon the state $|\psi(t)\rangle$ can—in general—be uniquely written as [14]

$$\widehat{H}|\psi(t)\rangle = \langle H\rangle|\psi(t)\rangle + \Delta H|\psi^\perp(t)\rangle, \tag{7}$$

where $\langle H\rangle = \langle\psi(t)|\widehat{H}|\psi(t)\rangle$, $\Delta H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$, and the orthogonal companion state $|\psi^\perp(t)\rangle$ belongs to the associated Hilbert subspace which is the orthogonal complement of the subspace containing $|\psi(t)\rangle$ and satisfies the conditions

$$\langle\psi^\perp(t)|\psi(t)\rangle = 0 \qquad \text{and} \qquad \Delta H = \langle\psi^\perp(t)|\widehat{H}|\psi(t)\rangle. \tag{8}$$

Equation (7) provides the following equivalent definition for the pointed weak energy when the dual form of this equation is first used to form the scalar product with the state $|\psi(0)\rangle$ and then this product is divided by $\langle\psi(t)|\psi(0)\rangle \neq 0$:

$$\frac{\langle\psi(t)|\widehat{H}|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle} = \langle H\rangle + \Delta H\frac{\langle\psi^\perp(t)|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle}. \tag{9}$$

When this is applied to the integrand of the exponent in equation (6), then the first term on the right-hand side of equation (9) identifies

$$\delta \equiv \frac{1}{\hbar}\int_0^\tau \langle H\rangle\,\mathrm{d}t \tag{10}$$

as a real valued dynamical phase which is acquired by the system as a result of the evolution of the state $|\psi(t)\rangle$ over the time interval $[0, \tau]$. The second term in equation (9) defines the complex valued pointed phase

$$\beta \equiv \frac{1}{\hbar}\int_0^\tau \Delta H\frac{\langle\psi^\perp(t)|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle}\,\mathrm{d}t \tag{11}$$

which is also acquired by the system as a result of the state's evolution over the time interval $[0, \tau]$.

In order to examine the geometric properties of the pointed phase, first observe that ΔH is invariant under the local $U(1)$ gauge transformation $|\psi(t)\rangle \to \mathrm{e}^{\mathrm{i}\theta(t)}|\psi(t)\rangle$ and that this transformation implies that $|\psi^\perp(t)\rangle \to \mathrm{e}^{\mathrm{i}\theta(t)}|\psi^\perp(t)\rangle$ (to see this simply note that if

$|\psi(t)\rangle \rightarrow \mathrm{e}^{\mathrm{i}\theta(t)}|\psi(t)\rangle$ and $\Delta H \neq 0$, then from equation (7): $\Delta H^{-1}(\widehat{H} - \langle H\rangle)|\psi(t)\rangle \rightarrow \mathrm{e}^{\mathrm{i}\theta(t)}\Delta H^{-1}(\widehat{H} - \langle H\rangle)|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\theta(t)}|\psi^{\perp}(t)\rangle$). Then, since

$$\frac{\langle\psi^{\perp}(t)|\,\mathrm{e}^{-\mathrm{i}\theta(t)}\,\mathrm{e}^{\mathrm{i}\theta(0)}|\psi(0)\rangle}{\langle\psi(t)|\,\mathrm{e}^{-\mathrm{i}\theta(t)}\,\mathrm{e}^{\mathrm{i}\theta(0)}|\psi(0)\rangle} = \frac{\langle\psi^{\perp}(t)|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle},$$

it can be concluded from equation (11) that the pointed phase β is invariant under local $U(1)$ gauge transformations.

Now consider the behaviour of equation (11) under the reparameterization $|\psi(t)\rangle = |\psi'(t'(t))\rangle \equiv |\psi'(t')\rangle$ over the interval $[t'(0), t'(\tau)]$ such that $t'(t)$ is monotone-increasing with state end-points $|\psi'(t'(0))\rangle = |\psi(0)\rangle$ and $|\psi'(t'(\tau))\rangle = |\psi(\tau)\rangle$. Using this parameterization, it is easily found that since

$$\widehat{H}|\psi(t)\rangle = \widehat{H}|\psi'(t')\rangle = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t'}|\psi'(t')\rangle\frac{\mathrm{d}t'}{\mathrm{d}t} = \widehat{H}'|\psi'(t')\rangle\frac{\mathrm{d}t'}{\mathrm{d}t}, \tag{12}$$

then

$$\langle H\rangle = \langle\psi'(t')|\widehat{H}'|\psi'(t')\rangle\frac{\mathrm{d}t'}{\mathrm{d}t} \equiv \langle H'\rangle\frac{\mathrm{d}t'}{\mathrm{d}t} \tag{13}$$

and

$$\langle H^2\rangle = \langle\psi'(t')|\widehat{H}'^2|\psi'(t')\rangle\left(\frac{\mathrm{d}t'}{\mathrm{d}t}\right)^2 \equiv \langle H'^2\rangle\left(\frac{\mathrm{d}t'}{\mathrm{d}t}\right)^2,$$

so that

$$\Delta H = \sqrt{\langle H'^2\rangle - \langle H'\rangle^2}\frac{\mathrm{d}t'}{\mathrm{d}t} \equiv \Delta H'\frac{\mathrm{d}t'}{\mathrm{d}t}, \tag{14}$$

i.e. $\Delta H\mathrm{d}t$ maps as a 1-form under this reparameterization. Furthermore, the reparameterization $|\psi(t)\rangle = |\psi'(t'(t))\rangle$ also implies the existence of a reparameterized orthogonal companion state such that $|\psi^{\perp}(t)\rangle = |\psi^{\perp\prime}(t'(t))\rangle$ (to see this rearrange equation (7) to find $|\psi^{\perp}(t)\rangle$, substitute $|\psi'(t'(t))\rangle$ for $|\psi(t)\rangle$, apply equations (12), (13) and (14), and require $\Delta H' \neq 0$). Thus,

$$\frac{1}{\hbar}\int_0^{\tau}\Delta H\frac{\langle\psi^{\perp}(t)|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle}\,\mathrm{d}t = \frac{1}{\hbar}\int_{t'(0)}^{t'(\tau)}\Delta H'\frac{\langle\psi^{\perp\prime}(t')|\psi'(t'(0))\rangle}{\langle\psi'(t')|\psi'(t'(0))\rangle}\,\mathrm{d}t'$$

from which it may be concluded that the pointed phase β is invariant under this reparameterization.

When taken together, these two invariance properties imply that β is a pure geometric phase [10] in the sense that its value depends only upon the associated smooth evolutionary path Γ in $\mathcal{P}$ and remains unchanged for any lifts to smooth monotone-increasing parameterized evolutionary paths γ in $\mathcal{H}$ such that $\Pi(\gamma) = \Gamma$. Therefore, equation (11) defines a geometric phase β—the *pointed geometric phase*. Note that this same conclusion is reached when the gauge and reparameterization invariance properties are analysed for the difference

$$\beta = \frac{1}{\hbar}\int_0^{\tau}W_0(t)\,\mathrm{d}t - \delta = -\mathrm{i}\int_0^{\tau}\frac{\left\langle\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}|\psi(0)\right\rangle}{\langle\psi(t)|\psi(0)\rangle}\,\mathrm{d}t + \mathrm{i}\int_0^{\tau}\left\langle\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}\middle|\psi(t)\right\rangle\mathrm{d}t.$$

In order to emphasize the geometric nature of β, this difference may also be expressed in the equivalent path integral form given by

$$\beta = -\mathrm{i}\int_{\Gamma}\frac{\langle\mathrm{d}\psi(t)|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle} + \mathrm{i}\int_{\Gamma}\langle\mathrm{d}\psi(t)|\psi(t)\rangle.$$

The reader is cautioned that this path integral representation for β employs a slight abuse of notation since each term in the sum is not individually gauge invariant.

4. Re β and the MS and AA geometric phases

Return now to the time integral of equation (9) and rewrite it as

$$\frac{1}{\hbar}\int_0^\tau W_0(t)\,\mathrm{d}t = \delta + \beta.$$

Since δ is real and $W_0(t)$ and β are both complex valued quantities, then the last equation implies that equation (6) can be equivalently written in (a non-polar) exponential form as

$$\langle\psi(\tau)|\psi(0)\rangle = \mathrm{e}^{\mathrm{i}(\delta+\mathrm{Re}\,\beta)-\mathrm{Im}\,\beta}, \tag{15}$$

where

$$\mathrm{Re}\,\beta = \frac{1}{\hbar}\int_0^\tau \mathrm{Re}\,W_0(t)\,\mathrm{d}t - \delta,$$

and

$$\mathrm{Im}\,\beta = \frac{1}{\hbar}\int_0^\tau \mathrm{Im}\,W_0(t)\,\mathrm{d}t.$$

Substitution of equations (2) and (3) for the integrands in the last two equations yields—upon integration—the following identities:

$$\mathrm{Re}\,\beta = \chi(\tau) - \delta \tag{16}$$

and

$$\mathrm{Im}\,\beta = \ln\frac{2}{\sqrt{4 - s^2(\tau)}}. \tag{17}$$

Here, use has been made of the fact that $\chi(0) = s(0) = 0$. Using equations (4) and (5) it is easily verified that—as required—both Re β and Im β are invariant under the aforementioned gauge and reparameterization transformations.

It may be concluded from equation (16) that since $\chi(\tau) = \delta + \mathrm{Re}\,\beta$, then the Pancharatnam phase associated with the arbitrary evolution of a state over a time interval $[0,\tau]$ is the sum of a dynamical phase which is defined by equation (10) and the real part of the pointed geometric phase which depends only upon the path traced in $\mathcal{P}$ by the evolving state during the interval $[0,\tau]$. Because of its geometric nature, it is interesting to determine the relationship between the real pointed geometric phase and the MS and AA geometric phases φ_{MS} and φ_{AA}, respectively. The former relationship can be established by comparing equation (16) with the expression for φ_{MS} for arbitrary evolutions given by equation (4) in [7]. There $\varphi_{\mathrm{MS}} = \chi_{\mathrm{MS}}(\tau) + \delta_{\mathrm{MS}}$, where $\chi_{\mathrm{MS}}(\tau) \equiv \arg\langle\psi(0)|\psi(\tau)\rangle$ and $\delta_{\mathrm{MS}} \equiv \frac{1}{\hbar}\int_0^\tau\langle H\rangle\,\mathrm{d}t$ are the associated Mukunda–Simon Pancharatnam and dynamical phases, respectively. This comparison readily reveals that $\chi(\tau) = -\chi_{\mathrm{MS}}(\tau)$ and $\delta = \delta_{\mathrm{MS}}$ so that

$$\mathrm{Re}\,\beta = \chi(\tau) - \delta = -\chi_{\mathrm{MS}}(\tau) - \delta_{\mathrm{MS}} = -(\chi_{\mathrm{MS}}(\tau) + \delta_{\mathrm{MS}}) = -\varphi_{\mathrm{MS}}.$$

The relationship between Re β and φ_{AA} can also be obtained in a similar manner by considering a cyclic evolution of a state such that $|\psi(\tau)\rangle = \mathrm{e}^{\mathrm{i}\phi}|\psi(0)\rangle$. In this case, $\chi(\tau) = -\phi$ so that equation (16) yields

$$\mathrm{Re}\,\beta = -\phi - \delta = -(\phi + \delta).$$

When the sum in parenthesis in this equation is compared with the expression for the AA geometric phase $\varphi_{\mathrm{AA}} = \phi_{\mathrm{AA}} + \delta_{\mathrm{AA}}$ for cyclic evolutions given by equation (3) in [3], it is found that $\phi = \phi_{\mathrm{AA}}$ and $\delta = \delta_{\mathrm{AA}}$. Here $\phi_{\mathrm{AA}} \equiv \arg\langle\psi(0)|\psi(\tau)\rangle$ and $\delta_{\mathrm{AA}} \equiv \frac{1}{\hbar}\int_0^\tau\langle H\rangle\,\mathrm{d}t$ are the associated Aharonov–Anandan Pancharatnam and dynamical phases, respectively. Consequently, the relationship between the real pointed geometric phase and φ_{AA} is

$$\mathrm{Re}\,\beta = -(\phi_{\mathrm{AA}} + \delta_{\mathrm{AA}}) = -\varphi_{\mathrm{AA}}.$$

Clearly, this same 'differing by a sign' relationship is required to hold for both the MS and AA phases since the AA phase is the MS phase for the special case that the evolution is cyclic.

5. The significance of Im β

Now consider the imaginary part of the pointed geometric phase. It can be inferred from equation (17) that since

$$e^{-\operatorname{Im}\beta} = e^{-\ln\frac{2}{\sqrt{4-s^2(\tau)}}} = \tfrac{1}{2}\sqrt{4-s^2(\tau)} = |\langle\psi(\tau)|\psi(0)\rangle|, \tag{18}$$

then the imaginary part of the pointed geometric phase defines the modulus of the correlation amplitude at time τ and therefore—in general—governs the survival probability $\Pr(\tau) \equiv |\langle\psi(\tau)|\psi(0)\rangle|^2 = e^{-2\operatorname{Im}\beta}$ of the initial state. For the special case that the evolution is cyclic, then $s(\tau) = s(0) = 0$ so that equation (17) becomes

$$\operatorname{Im}\beta = 0$$

and

$$\beta = \operatorname{Re}\beta = -\varphi_{\mathrm{AA}}. \tag{19}$$

Observe that equation (18) enables the complex valued exponential form for the correlation amplitude given by equation (15) to be equivalently expressed in the convenient polar form $\langle\psi(\tau)|\psi(0)\rangle = \frac{1}{2}\sqrt{4-s^2(\tau)}\,e^{i(\delta+\operatorname{Re}\beta)}$ with $\Pr(\tau) = 1 - \frac{1}{4}s^2(\tau)$.

Berry has identified a geometric amplitude factor for a quantum system undergoing a non-dissipative adiabatic evolution [15]. This factor is associated with the transition amplitude at $t \to +\infty$ for finding the system in a state different from the adiabatic eigenstate in which it was prepared at $t \to -\infty$. Although the probability for such a transition is exponentially weak and of order $e^{-\frac{1}{\epsilon}}$ (here ϵ is a small adiabatic parameter), for certain systems with time-dependent complex Hermitian Hamiltonians there exists an additional geometric exponential factor of unit order. For such two state systems with instantaneous eigenstates $|\nu_\pm(\epsilon t)\rangle$ it is found that this transition probability is to good approximation given by $\Pr(+\infty) \equiv |\langle\nu_-(+\infty)|\Psi(+\infty)\rangle|^2 \approx e^{-\Lambda_d}e^{\Lambda_g}$. Here the initial state of the system is assumed to be $|\nu_+(-\infty)\rangle$; $|\Psi(+\infty)\rangle$ is the associated evolved system state at $t \to +\infty$; $e^{-\Lambda_d}$ is a non-geometric dynamical factor of order $e^{-\frac{1}{\epsilon}}$ with Λ_d defined by equation (11) in [15], and e^{Λ_g} is a purely geometric factor with Λ_g defined by equation (13) in [15]. The exponent Λ_g derives its geometric character from its singular dependence upon the analytic continuation of the associated Hamiltonian's evolutionary path in its parameter space.

It is instructive to ascertain similarities that may exist between the geometric amplitude factor $e^{-\operatorname{Im}\beta}$ of $\langle\psi(\tau)|\psi(0)\rangle$ and $\operatorname{Im}\beta$ and Berry's geometric quantum amplitude factor and its concomitant geometric phase angle Λ_g. In particular, the observations that both $\Pr(\tau) = e^{-2\operatorname{Im}\beta}$ and $\Pr(+\infty) \approx e^{-\Lambda_d}e^{\Lambda_g} = e^{-(\Lambda_d-\Lambda_g)} \equiv e^{-\Theta}$ share a common exponential form and that $\operatorname{Im}\beta$ and Λ_g each possess certain geometric attributes suggest that these quantities may be more intimately related. However—as the following straightforward comparisons show—this is generally not the case: (i) $\langle\psi(\tau)|\psi(0)\rangle$ and $\langle\nu_-(+\infty)|\Psi(+\infty)\rangle$ have very different meanings, i.e. since $\Pr(\tau)$ is the survival probability for state $|\psi(0)\rangle$ at time $t = \tau$, whereas $\Pr(+\infty)$ is the contemporaneous probability at time $t \to +\infty$ that the system adiabatically transitions from the state $|\Psi(+\infty)\rangle$ to state $|\nu_-(+\infty)\rangle$ when the system was initially at time $t \to -\infty$ in state $|\nu_+(-\infty)\rangle$, then $\langle\Psi(+\infty)|\nu_+(-\infty)\rangle$ rather than $\langle\nu_-(+\infty)|\Psi(+\infty)\rangle$ serves as the associated survival probability amplitude analogue to $\langle\psi(\tau)|\psi(0)\rangle$; (ii) $\langle\nu_-(+\infty)|\Psi(+\infty)\rangle$ is defined strictly for an adiabatic evolution and is an adiabatic approximation, whereas $\langle\psi(\tau)|\psi(0)\rangle$ is exactly valid and is defined for an arbitrary evolution in which $|\psi\rangle$ need not be an eigenstate of the associated system Hamiltonian; (iii) the exponent $2\operatorname{Im}\beta$ is geometric and does not contain a dynamical term, whereas Θ does and is the difference $\Lambda_d - \Lambda_g$, and (iv) the geometric nature of $\operatorname{Im}\beta$ follows from its association with an evolutionary path in projective Hilbert space, whereas that of Λ_g is derived from the analytic continuation of a Hamiltonian curve in its parameter space. It can be concluded from these comparisons that—not only are $\operatorname{Im}\beta$ and Λ_g geometrically defined differently—but also that $\operatorname{Im}\beta$ specifies a geometric amplitude factor $e^{-\operatorname{Im}\beta}$ of $\langle\psi(\tau)|\psi(0)\rangle$ that—in general—is distinct from the Berry geometric amplitude factor that is associated with $\langle\nu_-(+\infty)|\Psi(+\infty)\rangle$.

6. Rate of change relationships for the pointed geometric phase

The concise representation for β given by equation (11) is useful for describing the rates of change of $\operatorname{Re}\beta$ and $\operatorname{Im}\beta$ with respect to the Fubini–Study distance. In particular, using equations (4) and (5), along with the fact that $\hbar^{-1}\Delta H\,dt = 2^{-1}ds$ [16], in equation (11) yields

$$d\beta = \frac{1}{\hbar}\Delta H\frac{\langle\psi^{\perp}(t)|\psi(0)\rangle}{\langle\psi(t)|\psi(0)\rangle}\,dt = \frac{1}{2}\sqrt{\frac{4-s_{\perp}^2}{4-s^2}}e^{i(\chi_{\perp}-\chi)}\,ds,$$

$s \neq 2$, or

$$\frac{d\beta}{ds} = \left|\frac{d\beta}{ds}\right| e^{i(\chi_{\perp}-\chi)},$$

where

$$\left|\frac{d\beta}{ds}\right| = \frac{1}{2}\sqrt{\frac{4-s_{\perp}^2}{4-s^2}} \tag{20}$$

is the modulus phase s-speed. Here, s and $s_{\perp}$ are the Fubini–Study metric distances from $|\psi(t)\rangle$ to $|\psi(0)\rangle$ and from $|\psi^{\perp}(t)\rangle$ to $|\psi(0)\rangle$, respectively, and χ and $\chi_{\perp}$ are the associated Pancharatnam phases. The associated real and imaginary phase s-speeds are defined by

$$\frac{d\operatorname{Re}\beta}{ds} = \left|\frac{d\beta}{ds}\right|\cos(\chi_{\perp}-\chi) \tag{21}$$

and

$$\frac{d\operatorname{Im}\beta}{ds} = \left|\frac{d\beta}{ds}\right|\sin(\chi_{\perp}-\chi), \tag{22}$$

respectively.

Although the real and imaginary phase s-speeds depend upon both s and $s_{\perp}$ (via the modulus phase s-speed) and the difference $\chi_{\perp}-\chi$ (via the argument of trigonometric co-functions), their ratio depends only upon the Pancharatnam phase difference. In particular, dividing equation (21) by equation (22) provides the following s-speed ratio which describes how a change in $\operatorname{Re}\beta$ relates to a change in $\operatorname{Im}\beta$:

$$\frac{d\operatorname{Re}\beta}{d\operatorname{Im}\beta} = \cot(\chi_{\perp}-\chi). \tag{23}$$

(Obviously, an analogous s-speed ratio describing the change in $\operatorname{Im}\beta$ due to a change in $\operatorname{Re}\beta$ is obtained from the reciprocal of this equation.) This expression shows—for example—that for first quadrant Pancharatnam phase differences, the greatest change in $\operatorname{Re}\beta$ associated with a change in $\operatorname{Im}\beta$ occurs for small phase differences.

7. An example: the pointed geometric phase for a cyclic evolution

The purpose of this section is to illustrate aspects of the theory developed above by first using equation (11) to calculate the pointed geometric phase β acquired after one period of a cyclic evolution induced by the action of a uniform magnetic field upon a spin-$\frac{1}{2}$ particle with magnetic moment μ. If this uniform magnetic field $\mathbf{B}$ is oriented along the z-axis of a three-dimensional Cartesian reference frame, then the interaction Hamiltonian in this rest frame is $\widehat{H} = -\mu B\widehat{\sigma}_z$, where $\widehat{\sigma}_z$ is a Pauli spin operator which acts upon the associated orthogonal spin basis eigenkets $|\pm\rangle$ according to the rule $\widehat{\sigma}_z|\pm\rangle = \pm|\pm\rangle$. If the spin direction is fixed at an angle θ with respect to the positive z-axis, then at any time t the normalized system state is given by

$$|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\alpha t}\cos\frac{\theta}{2}|+\rangle + \mathrm{e}^{-\mathrm{i}\alpha t}\sin\frac{\theta}{2}|-\rangle,$$

where $\alpha = \frac{\mu B}{\hbar}$. It is clear from this that the evolution of $|\psi(t)\rangle$ is periodic with period $\tau = \frac{\pi\hbar}{\mu B}$ and

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$$

so that

$$\langle\psi(t)|\psi(0)\rangle = \cos\alpha t - \mathrm{i}\cos\theta\sin\alpha t \tag{24}$$

and

$$s^2(t) = 4\sin^2\alpha t\sin^2\theta.$$

It is also readily determined that

$$\langle H\rangle = -\mu B\langle\sigma_z\rangle = -\mu B\cos\theta$$

and

$$\langle H^2\rangle = \mu^2B^2\langle\sigma_z^2\rangle = \mu^2B^2$$

so that

$$\Delta H = \mu B\sin\theta. \tag{25}$$

In order to calculate β using equation (11) it is necessary to select the orthogonal companion state $|\psi^\perp(t)\rangle$ that satisfies conditions (8). Straightforward inspection shows that

$$|\psi^\perp(t)\rangle = -\mathrm{e}^{\mathrm{i}\alpha t}\sin\frac{\theta}{2}|+\rangle + \mathrm{e}^{-\mathrm{i}\alpha t}\cos\frac{\theta}{2}|-\rangle$$

satisfies these conditions and that

$$\langle\psi^\perp(t)|\psi(0)\rangle = \mathrm{i}\sin\theta\sin\alpha t. \tag{26}$$

Using this along with equations (24) and (25) in equation (11) yields

$$\operatorname{Im}\beta = \frac{\mu B\sin^2\theta}{\hbar}\int_0^{\frac{\pi\hbar}{\mu B}}\left[\frac{\sin\alpha t\cos\alpha t}{1-\sin^2\theta\sin^2\alpha t}\right]\mathrm{d}t = 0$$

and [17]

$$\begin{aligned}\operatorname{Re}\beta &= -\frac{\mu B\sin^2\theta\cos\theta}{\hbar}\int_0^{\frac{\pi\hbar}{\mu B}}\left[\frac{\sin^2\alpha t}{\cos^2\alpha t+\cos^2\theta\sin^2\alpha t}\right]\mathrm{d}t\\ &= [\alpha t\cos\theta - \tan^{-1}(\cos\theta\tan\alpha t)]_{t=0}^{t=\frac{\pi\hbar}{\mu B}}\\ &= \pi\cos\theta - \pi\\ &= -(\pi - \pi\cos\theta),\end{aligned}$$

where use is made of the complex conjugate of equation (24) and the fact that $\arg\langle\psi(0)|\psi(t)\rangle = \tan^{-1}(\cos\theta\tan\alpha t)$. Thus—as required—$\beta = \mathrm{Re}\,\beta = -(\phi_{\mathrm{AA}} + \delta_{\mathrm{AA}}) = -\varphi_{\mathrm{AA}}$ and equation (19) is satisfied.

Now consider the s-speeds associated with this system as it evolves with time. It is easily deduced from equation (26) that $\chi_\perp = \frac{\pi}{2}$ (because $\mathrm{e}^{\mathrm{i}\chi_\perp} = \langle\psi^\perp(t)|\psi(0)\rangle|\langle\psi^\perp(t)|\psi(0)\rangle|^{-1} = \mathrm{i}$). Using this and the right triangle defined by equation (24) which has χ as the fourth quadrant reference angle, $\cos\alpha t$ as the length of the side adjacent to χ and $\cos\theta\sin\alpha t$ as the length of the side opposite to χ, it is found that

$$\cos(\chi_\perp - \chi) = \sin\chi = -\frac{\cos\theta\sin\alpha t}{\sqrt{1-\sin^2\theta\sin^2\alpha t}}$$

and

$$\sin(\chi_\perp - \chi) = \cos\chi = \frac{\cos\alpha t}{\sqrt{1-\sin^2\theta\sin^2\alpha t}}.$$

Substitution of these results along with the fact that (from equation (20))

$$\left|\frac{\mathrm{d}\beta}{\mathrm{d}s}\right| = \frac{\sin\theta\sin\alpha t}{2\sqrt{1-\sin^2\theta\sin^2\alpha t}}$$

into equations (21) and (22) yields the time-dependent phase s-speed expressions

$$\frac{\mathrm{d}\,\mathrm{Re}\,\beta}{\mathrm{d}s} = -\frac{\sin\theta\cos\theta\sin^2\alpha t}{2(1-\sin^2\theta\sin^2\alpha t)}$$

and

$$\frac{\mathrm{d}\,\mathrm{Im}\,\beta}{\mathrm{d}s} = \frac{\sin\theta\sin\alpha t\cos\alpha t}{2(1-\sin^2\theta\sin^2\alpha t)}.$$

Thus, during one cycle with period τ and $0 < \theta < \frac{\pi}{2}$, $\frac{\mathrm{d}\,\mathrm{Re}\,\beta}{\mathrm{d}s} \leqslant 0$ when $0 \leqslant \alpha t \leqslant \pi$ so that the associated phase accumulates negatively with changing s during a cycle. In contrast to this, $\frac{\mathrm{d}\,\mathrm{Im}\,\beta}{\mathrm{d}s} \geqslant 0$ when $0 \leqslant \alpha t \leqslant \frac{\pi}{2}$, whereas $\frac{\mathrm{d}\,\mathrm{Im}\,\beta}{\mathrm{d}s} \leqslant 0$ when $\frac{\pi}{2} < \alpha t \leqslant \pi$. In this case, the survival probability for the initial state diminishes until $t = \frac{\tau}{2}$, after which it increases to its unit value at $t = \tau$.

Also, observe that dividing the first of the last two equations by the second yields the associated s-speed ratio

$$\frac{\mathrm{d}\,\mathrm{Re}\,\beta}{\mathrm{d}\,\mathrm{Im}\,\beta} = -\cos\theta\tan\alpha t = \tan\chi.$$

Here use has been made of equation (24) and the fact that $\tan\chi = \frac{\mathrm{Im}\,\langle\psi(t)|\psi(0)\rangle}{\mathrm{Re}\,\langle\psi(t)|\psi(0)\rangle}$. Hence, this result is that of equation (23) when $\chi_\perp = \frac{\pi}{2}$ and it is easily seen from this that—as required for first quadrant Pancharatnam phase differences—the value of this derivative is greatest when $\chi \approx \frac{\pi}{2}$.

8. Closing remarks

Pointed weak energy has been used to identify a complex valued geometric phase β—the *pointed geometric phase*—that is associated with an arbitrary evolution of a quantum state over a time interval $[0, \tau]$. This phase can be computed directly from a new and concise representation expressed as a time integral which depends upon the energy uncertainty, the associated evolving state, and its orthogonal companion state. Several new rate of change results have been obtained from this representation. Two of these results are phase s-speed expressions which describe how the real and imaginary parts of the pointed geometric phase vary with the Pancharatnam phases and the Fubini–Study metric distances of the evolving state and its orthogonal companion relative to the initial state. An associated s-speed ratio has also been obtained which provides a simple Pancharatnam phase-dependent relationship between changes in the real and imaginary parts of the pointed geometric phase.

The real and imaginary parts of the pointed geometric phase exhibit several interesting properties. In particular, the negative real part of the pointed geometric phase has been shown to be identical to the MS geometric phase for arbitrary evolutions (and consequently to the AA geometric phase for cyclic evolutions) and the imaginary part of the pointed geometric phase has been shown to (i) govern the survival probability of the evolving state; (ii) provide for a simple polar form representation of the correlation amplitude that is expressed in terms of the Fubini–Study metric distance, the dynamical phase, and Re β, and (iii) differ from the geometric phase concomitant with Berry's geometric quantum amplitude factor. When taken together, these properties suggest that β defines a new complex valued geometric phase.

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